Compendium: Solutions to selected exercises from John David Jackson: Classical Electrodynamics (3rd edition)

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Chapter 1

From the first chapter the exercises 1.1, 1.5, 1.6 and 1.14 are solved.

Exercise **1.1** Gauss's Law

Use Gauss's theorem [and (1.21) if necessary] to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0 where σ is the charge density per unit area on the surface.

(a) As charges per definition can move freely in a conductor, no electric field can be situated there, or the charges would be subject to a force, by $\vec{F} = q\vec{E}$. Hence the electric field must vanish in the conductor. Gauss's law states that:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{1}$$

and having $\vec{E} = 0$, ρ must also be zero. As excess charge cannot be situated *in* the conductor, it must lie entirely on its surface.

(b) As the net excess charge density resides on the surface of the conductor, a field will arise outside the conductor based on the configuration of the charge. The field outside the conductor, arising from charges placed inside it, can be found using Gauss's law in integral form:

$$\oint_S \vec{E} \cdot \vec{n}\, da = \sum_i \frac{q_i}{\epsilon_0}, \tag{2}$$

where q_i is *all* charges contained within the conductor.

(c) If the electric field at the surface of the conductor has any component parallel to the surface E_x, the charges on the surface will be subject to a force $F_x = qE_x$, which will move the charge around the surface until the electric field is aligned normal to the surface. This is thus the direction of the electric field (at least in electrostatics...).

To calculate the magnitude of the field, consider a Gaussian pillbox, small enough to be aligned parallel to the conductor surface. As the electric field is zero within the conductor and the direction of the electric field is normal to the surface area (and thus only the top of the pillbox contributes), Gauss's law becomes:

$$\oint_S \vec{E} \cdot \vec{n}\; da = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) d^3x \Leftrightarrow \oint_S E = \frac{1}{\epsilon_0} \int_A \rho \vec{x} d^2 x \Leftrightarrow A_{top} E = \frac{Q_{enc}}{\epsilon_0}, \tag{3}$$

where Q_{enc} is the charge enclosed in the pillbox and A_{top} is the top area. Setting $\sigma = \frac{Q_{enc}}{A_{top}}$ as the charge density per unit area, it is seen that $E = \frac{\sigma}{\epsilon_0}$ as desired.

Exercise **1.5** Potential of the Hydrogen Atom

The time-averaged potential of a neutral hydrogen atom is given by:

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right), \tag{4}$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

The charge distribution can be calculated from the potential by means of the Poisson equation:

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \tag{5}$$

where the radial part of the Laplace operator ∇^2 in spherical coordinates is (suppressing the non-radial terms, as the potential is spherically symmetric):

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} (\psi). \tag{6}$$

The differentiation is carried out using the ordinary rules:

$$\begin{aligned} -\frac{4\pi\rho}{q} &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left[\frac{e^{-\alpha r}}{r} + \frac{\alpha e^{-\alpha r}}{2}\right] = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left[-\frac{\alpha e^{-\alpha r}}{r} + e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\alpha^2 e^{-\alpha r}}{2}\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[-\alpha e^{-\alpha r} r + e^{-\alpha r} r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\alpha^2 r^2 e^{-\alpha r}}{2}\right] \\ &= \frac{1}{r^2}\left[\alpha^2 e^{-\alpha r} r - \alpha e^{-\alpha r} + e^{-\alpha r}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \alpha e^{-\alpha r} - \alpha^2 r e^{-\alpha r} - \frac{r^2\alpha^2 e^{-\alpha r}}{2}\right] \\ \Leftrightarrow \rho &= \frac{q e^{-\alpha r}}{4\pi}\left[\frac{\alpha^3}{2} - \nabla^2\left(\frac{1}{r}\right)\right]. \end{aligned} \tag{7}$$

The Laplacian of $\frac{1}{r}$ has a singular nature and the identity from Jackson equation (1.31) is used:

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(r). \tag{8}$$

The factor of $e^{-\alpha r}$ vanishes when multiplied with $\delta(r)$, and the final result is:

$$\rho(r) = \left[\frac{\alpha^3 e^{-\alpha r}}{8\pi} + \delta(r) \right] q. \tag{9}$$

This charge distribution is physically interpreted as a sharp, discrete peak in the center (the electron), and a continuous distribution (electron cloud) vanishing as $r \to \infty$. This is fine as a classical description, but the hydrogen atom must of course be described with quantum mechanics.

Exercise **1.6** Capacitance

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of:

(a) Two large, flat conducting sheets of area A, separated by a small distance d.

(b) Two concentric conducting spheres with radii a, b $(b > a)$.

(c) Two concentric conducting cylinders of length L, large compared to their radii a, b $(b > a)$.

(d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

The capacitance is the quantity $C \equiv \frac{Q}{\Phi}$, and will here be calculated for three geometric configurations.

(a) Parallel plates Following the same idea as when constructing a Gaussian pillbox in equation (3), now both the top and the bottom contributes, and clearly the field from one plate is:

$$\vec{E} = \frac{Q}{2A\epsilon_0}\hat{n}, \tag{10}$$

where $\hat{n}$ is the unit vector pointing away from the surface. Because of the symmetry of the parallel plate capacitor, the magnitude of the electric field between the plates is:

$$E = \frac{Q}{A\epsilon_0}. \tag{11}$$

The potential between the plates is therefore:

$$\Phi = \int_0^d E\ dl = \frac{Qd}{A\epsilon_0}, \tag{12}$$

making the capacitance:

$$C = \frac{Q}{\Phi} = \frac{A\epsilon_0}{d}. \tag{13}$$

(b) Concentric spheres The Gaussian surface is drawn as a sphere with $a < r < b$. Using Gauss's law in integral form, this symmetry can be used:

$$\oint_S \vec{E} \cdot \vec{n}\ da = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) d^3x \Leftrightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}, \tag{14}$$

where $\hat{r}$ is the unit vector in the radial direction. From the electric field the electrostatic potential is found:

$$\Phi = -\int \vec{E} d\vec{l} = -\int_b^a \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}\ dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right). \tag{15}$$

Hence the capacitance is:

$$C = \frac{Q}{\Phi} = 4\pi\epsilon_0 \frac{ab}{b-a}. \tag{16}$$

(c) Concentric cylinders The Gaussian surface is drawn as a cylinder with $a < r < b$. Using Gauss's law in integral form, this symmetry can be used:

$$\oint_S \vec{E} \cdot \vec{n}\ da = \frac{1}{\epsilon_0} \int_V \rho(\vec{x})\ d^3x \Leftrightarrow \vec{E} = \frac{Q}{2\pi r L \epsilon_0} \hat{r}. \tag{17}$$

The electrostatic potential is:

$$\Phi = -\int \vec{E}\ d\vec{l} = \frac{Q}{2\pi L \epsilon_0} \int_b^a \frac{1}{r}\ dr = \frac{Q}{2\pi L \epsilon_0} \log(b/a). \tag{18}$$

And the capacitance is:

$$C = \frac{Q}{\Phi} = \frac{2\pi L \epsilon_0}{\log(b/a)}. \tag{19}$$

(d) Using the result from **(c)**, the diameter of the outer conductor is:

$$b = a \cdot \exp\left(\frac{2\pi\epsilon_0 L}{C} \right). \tag{20}$$

Plugging in the numbers $a = 1$ mm, $\epsilon_0 = 8.9 \times 10^{-12}$ F/m and $C_1 = 3 \times 10^{-11}$ F/m, $C_2 = 3 \times 10^{-12}$ F/m, one obtains:

$$b_1 = 6 \text{ mm}\ ,\ b_2 = 1.2 \times 10^8 \text{ mm}. \tag{21}$$

Exercise **1.14** Electrostatic Green Functions

Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\vec{y}$ and $\phi = G(\vec{x},\vec{y}), \psi = G(\vec{x}',\vec{y})$, with $\nabla_y^2 G(\vec{z},\vec{y}) = -4\pi\delta(\vec{y}-\vec{z})$. Find an expression for the difference $[G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})]$ in terms of an integral over the boundary surface S.

(a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\vec{x},\vec{x}')$ must be symmetric in $\vec{x}$ and $\vec{x}'$.

(b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\vec{x},\vec{x}')$ to show that $G_N(\vec{x},\vec{x}')$ is not symmetric in general, but that $G_N(\vec{x},\vec{x}') - F(x)$ is symmetric in $\vec{x}$ and $\vec{x}'$ where:

$$F(x) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{y}) da_y. \tag{22}$$

(c) Show that the addition of $F(\vec{x})$ to the Green function does not affect the potential $\Phi(\vec{x})$. See problem 3.23 for an example of the Neumann Green function.

Green's theorem (Jackson (1.35)) is:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\ d^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\psi}{\partial n}\right]\ da. \tag{23}$$

Equation 23 is rewritten using the integration variable $\vec{y}$ and:

$$\phi = G(\vec{x},\vec{y})\ ,\ \psi = G(\vec{x}',\vec{y})\ \text{with: } \nabla_y^2 G(\vec{z},\vec{y}) = -4\pi\delta(\vec{y}-\vec{z}).$$

In this way the difference $[G(\vec{x}-\vec{x}') - G(\vec{x}'-\vec{x})]$ is found:

$$\begin{aligned}
\int_V G(\vec{x},\vec{y})\nabla_y^2 G(\vec{x}',\vec{y}) - G(\vec{x}',\vec{y})\nabla_y^2 G(\vec{x},\vec{y})\ d^3y &= \oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n}\right] da \Leftrightarrow \\
4\pi\int_V G(\vec{x}',\vec{y})\delta(\vec{y}-\vec{x}) - G(\vec{x},\vec{y})\delta(\vec{y}-\vec{x}')d^3y &= \oint_S \ldots\ da \Leftrightarrow \\
4\pi\left[G(\vec{x}',\vec{x}) - G(\vec{x}-\vec{x}')\right] &= \oint_S \ldots\ da \Leftrightarrow \\
[G(\vec{x}-\vec{x}') - G(\vec{x}'-\vec{x})] &= -\frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n}\right] da
\end{aligned}$$

This identity is used to investigate symmetry properties of $G(\vec{x}',\vec{x})$ as G is symmetric in exchange of $\vec{x}$ and $\vec{x}'$ *iff*:

$$[G(\vec{x}-\vec{x}') - G(\vec{x}'-\vec{x})] = -\frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n}\right] da = 0. \tag{24}$$

(a) Dirichlet boundary conditions. The Dirichlet boundary condition states:

$$G_D(\vec{x}, \vec{x}') = 0 \text{ for } \vec{x}' \text{ on } S. \tag{25}$$

Using this condition with equation (24), it is readily obtained that $G_D(\vec{x}, \vec{x}')$ is symmetric in exchange of $\vec{x}$ and $\vec{x}'$ as:

$$G(\vec{x} - \vec{x}') - G(\vec{x}' - \vec{x}) = -\frac{1}{4\pi} \oint_S \left[G(\vec{x}, \vec{y}) \frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial n} \right] da, \tag{26}$$

since $G_D(\vec{x}, \vec{y}) = 0$ on the surface, the surface integral vanishes, and:

$$G(\vec{x} - \vec{x}') - G(\vec{x}' - \vec{x}) = 0. \tag{27}$$

(b) Neumann boundary conditions. The Neumann boundary condition states:

$$\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}') = -\frac{4\pi}{S} \text{ for } \vec{x}' \text{ on } S. \tag{28}$$

The identity from equation (24) becomes:

$$G(\vec{x} - \vec{x}') - G(\vec{x}' - \vec{x}) = -\frac{1}{S} \oint_S G_N(\vec{x}', \vec{y}) - G_N(\vec{x}, \vec{y}) \; da, \tag{29}$$

which is not zero in general, and $G_N(\vec{x}, \vec{x}')$ is not generally symmetric. To impose symmetry the following derived quantities can be used:

$$\begin{aligned} G_{N'}(\vec{x}, \vec{x}') &= G_N(\vec{x}, \vec{x}') - \frac{1}{S} \oint_S G_N(\vec{x}, \vec{y}) \; da \text{ , and} \\ G_{N'}(\vec{x}', \vec{x}) &= G_N(\vec{x}', \vec{x}) - \frac{1}{S} \oint_S G_N(\vec{x}', \vec{y}) \; da. \end{aligned} \tag{30}$$

Then:

$$\begin{aligned} &G_{N'}(\vec{x}, \vec{x}') - G_{N'}(\vec{x}', \vec{x}) = \\ &G_N(\vec{x}, \vec{x}') - G_N(\vec{x}', \vec{x}) - \frac{1}{S} \oint_S \left[G_N(\vec{x}, \vec{y}) + G_N(\vec{x}', \vec{y}) \right] \; da, \end{aligned} \tag{31}$$

and the non-zero right side in equation (29) vanishes.

(c). The solution to the Poisson equation for Neumann boundary conditions is (Jackson equation 1.46):

$$\Phi(\vec{x}) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G_N(\vec{x}, \vec{x}') \; d^3x' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} G_N \; da'. \tag{32}$$

Adding F to G_N makes equation (32):

$$\begin{aligned}\Phi' =& \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')(G+F)\ d^3x + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}(G+F)\ da' + \langle\Phi\rangle_S = \\ & \frac{1}{4\pi\epsilon_0}\left[\int_V \rho(\vec{x}')G\ d^3x' + \int_V \rho(\vec{x}')F\ d^3x'\right] + \frac{1}{4\pi}\left[\oint_S \frac{\partial\Phi}{\partial n'}G\ da' + \oint_S \frac{\partial\Phi}{\partial n'}F\ da'\right] + \langle\Phi\rangle_S = \\ & \Phi(\vec{x}) + \frac{F}{4\pi}\underbrace{\left[\int_V \frac{\rho(\vec{x}')}{\epsilon_0}\ d^3x' + \oint_S \frac{\partial\Phi}{\partial n'}\right]}_{\text{Should be zero.}} + \langle\Phi\rangle_S\,.\end{aligned} \tag{33}$$

For the potential to be unaffected by F, the term within the braces should be zero. This is readily shown using Gauss's law in integral form and the definition of the potential:

$$\begin{aligned}\int_V \frac{\rho(\vec{x}')}{\epsilon_0}\ d^3x + \oint_S \frac{\partial\Phi}{\partial n'} = \oint_S \vec{E}\cdot\vec{n}\ da + \oint_S \frac{\partial\Phi}{\partial n'} = \\ -\oint_S \vec{\nabla}\Phi\cdot\vec{n}\ da + \oint_S \frac{\partial\Phi}{\partial n'} = -\oint_S \frac{\partial\Phi}{\partial n} + \oint_S \frac{\partial\Phi}{\partial n'} = 0\end{aligned} \tag{34}$$

Chapter 4

From the fourth chapter the exercises 4.1 and 4.4 are solved.

Exercise **4.1** Multipole expansion

Calculate the multipole moments q_{lm} of the charge distributions shown as parts **(a)** and **(b)** (figure 1). Try to obtain results for the non vanishing moments valid for all l, but in each case find the first *two* sets of non vanishing moments at the very least.

(c) For the charge distribution of the second set **(b)** write down the multipole expansion for the potential. Keeping only the lowest-order terms in the expansion, plot the potential in the $x-y$ plane as a function of distance from the origin for distances greater than a.

(d) Calculate directly from Coulomb's law the exact potential for **(b)** in the $x-y$ plane. Plot it as a function of distance and compare with the result found in part **(c)**. Divide out the asymptotic form in parts **(c)** and **(d)** to see the behavior at large distances more clearly.

(a) The charge distribution is:

$$\rho(r,\phi,\theta) = \frac{q}{a^2}\delta(r-a)\delta(\cos(\theta))[\delta(\phi)+\delta(\phi-\frac{\pi}{2})-\delta(\phi-\pi)-\delta(\phi-\frac{3\pi}{2})]. \quad (35)$$

The multipole moment is given by:

$$q_{lm} = \int Y^*_{lm}(\theta,\phi) r^l \rho(\vec{x})\ d^3x. \quad (36)$$

In spherical coordinates the integral becomes:

$$q_{lm} = \frac{q}{a^2}\iiint Y^*_{lm}(\theta,\phi) r^{l+2}\delta(r-a)\delta(\cos(\theta))\sin(\theta)\cdot \\ [\delta(\phi)+\delta(\phi-\frac{\pi}{2})-\delta(\phi-\pi)-\delta(\phi-\frac{3\pi}{2})]\ drd\phi d\theta. \quad (37)$$

The spherical harmonic function can be decomposed (Jackson equation (3.53)) as:

$$Y^*_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\theta))e^{-im\phi}. \quad (38)$$

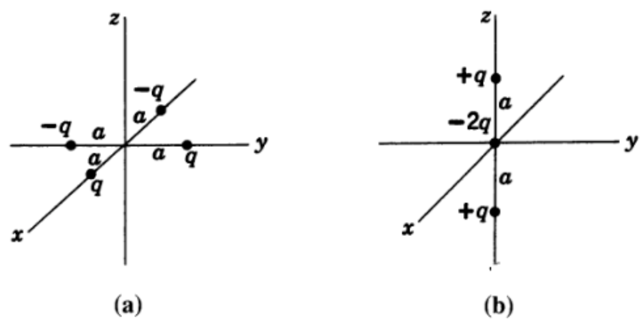

Figure 1: Discrete charge distributions for exercise 4.1.

The integral can then be carried out, here split up in it's partial terms:

$$\begin{aligned} &\int \delta(r-a) r^{l+2}\, da = a^{l+2}; \\ &\int P_l^m(\cos(\theta)\delta(\cos(\theta)))\, d\theta = P_l^m(0); \\ &\int e^{-im\phi}[\delta(\phi) + \delta(\phi - \frac{\pi}{2}) - \delta(\phi - \pi) - \delta(\phi - \frac{3\pi}{2})]\, d\phi = 1 + (-i)^m - (-1)^m - i^m = \\ &\quad 1 + (-1)^m (i)^m - (-1)^m - i^m = (1 - i^m) - (-1)^m (1 - i^m) = (1 - i^m)[1 - (-1)^m]. \end{aligned} \tag{39}$$

The last part is readily seen to be $2(1-i^m)$ for m odd and zero for m even. This makes the full expression:

$$q_{lm} = 2qa^l \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(0)(1 - i^m) \text{ for } m \text{ odd, and 0 for } m \text{ even.} \tag{40}$$

The expression in equation (40) can be used to calculate moments for specific values of l. As the exercise asks explicitly for the two first non vanishing ones, these are calculated (using Mathematica). The monopole moment ($l = 0$) vanishes as m is zero (which also makes sense since the net charge is zero). The dipole moments are:

$$q_{1,-1} = qa\,(1+i)\sqrt{\frac{1}{\pi}}\ ,\ q_{1,0} = 0\ ,\ q_{1,1} = 2qa(1-i)\sqrt{\frac{3}{8\pi}}. \tag{41}$$

The quadrupole moments all vanish. Some because m is even, the others because

$P_l^m(0) = 0$ in those cases. The non-zero octupole moments are:

$$q_{3,-3} = \frac{1}{24}a^3 q(1-i)\sqrt{\frac{363}{2\pi}}\ ,\ q_{3,-1} = -\frac{1}{4}a^3 q(1+i)\sqrt{\frac{15}{4\pi}},$$
$$q_{3,1} = a^3 q(1-i)\sqrt{\frac{3}{4\pi}}\ ,\ q_{3,3} = -a^3 q(1+i)\sqrt{\frac{35}{16\pi}}. \tag{42}$$

(b) The charge distribution is this time split in two terms, where the one describing the $-2q$ situated at the origin, must clearly be dependent of $\frac{1}{r^2}$:

$$\rho(r,\phi,\theta) = \frac{q}{2\pi a^2}\delta(r-a)[\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1)] - \frac{q}{2\pi r^2}\delta(r). \tag{43}$$

The azimuthal symmetry of the problem simplifies the expansion of the spherical harmonic function (Jackson equation (3.57)) greatly, and the multipole moment becomes:

$$q_{lm} = \frac{q}{2\pi}\sqrt{\frac{2l+1}{4\pi}}\int_{u=-1}^{u=1}\int_0^a\int_0^{2\pi}\frac{r^{l+2}}{a^2}\delta(r-a)[\delta(u+1)+\delta(u-1)]P_l(u) - \delta(r)r^l P_l(u)\ d\phi dr du \tag{44}$$

where the integration variable θ is substituted by $u = \cos(\theta) \Rightarrow d\theta = -\frac{du}{\sin(\theta)}$. The r and ϕ parts integrates easily, and:

$$q_{lm} = qa^l\sqrt{\frac{2l+1}{4\pi}}\left[\int_{-1}^{1}(\delta(u+1)+\delta(u-1))P_l(u)\ du - \int_{-1}^{1}P_l(u)\ du\right]. \tag{45}$$

The first integral evaluates to: $(P_l(1)+P_l(-1)) = (1+(-1)^l)$, using that $P_l(1) = 1$ for all l, and since $P_l(-x) = (-1)^l P_l(x)$, ($P_l(x)$ is either even or odd), $P_l(-1) = (-1)^l$.

For the second integral, a result regarding integrals of Legendre polynomials is used (not in Jackson):

$$\int_{-1}^{1} f(x)P_n(x)\ dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1)^n\frac{d^n f}{dx^n}dx. \tag{46}$$

Using $f(x) = 1$, the second integral evaluates to 2 when $l = 0$ and 0 otherwise. Collecting the terms, the multipole moment valid for all values of l is:

$$q_{lm} = qa^l\sqrt{\frac{2l+1}{4\pi}}\left([1+(-1)^l] - 2\delta(l)\right)\ \text{if } m = 0,\ 0 \text{ otherwise.} \tag{47}$$

Looking at the charge distribution can provide some (physical) insight to this expression. As the net charge is zero, clearly $q_{00} = 0$, and the second term serves to provide exactly this result. Realizing this, equation (47) can be rewritten in a somewhat simpler form:

$$q_{lm} = qa^l \sqrt{\frac{2l+1}{4\pi}} \text{ if } m = 0 \text{ and } l > 0 \text{ is even, 0 otherwise.} \tag{48}$$

The first two non vanishing moments is thus seen to be the $l = 2$ and $l = 4$ terms:

$$q_{20} = \sqrt{\frac{5}{\pi}} a^2 q \text{ and } q_{40} = \sqrt{\frac{9}{\pi}} a^4 q. \tag{49}$$

(c) The full expression for the multipole expansion of the potential is (Jackson equation 4.1):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}. \tag{50}$$

The expression for q_{lm} from equation (48) is used and, remembering the azimuthal symmetry, the second sum vanishes, and the spherical harmonics is expanded in Legendre polynomials:

$$\Phi(r, \cos(\theta)) = \frac{q}{4\pi\epsilon_0 a} \sum_{l=2,\text{ even}}^{\infty} P_l(\cos(\theta)) \left(\frac{a}{r}\right)^{l+1}. \tag{51}$$

The first term of the expansion in the $x-y$ plane (using $P_2(\cos(\theta)) = \frac{1}{4}(1+3\cos(2\theta))$) is:

$$\Phi(r) = -\frac{qa^2}{4\pi\epsilon_0} \frac{1}{r^3}. \tag{52}$$

This is plotted in figure 2 (left), the potential in units of $-\frac{q}{4\pi\epsilon_0 a}$.

(d) Using Coulomb's law directly, the potential carries a term for each charge, proportional to the magnitude of the charge divided by the distance to the charge. Hence:

$$\Phi_C(r) = -\frac{q}{2\pi\epsilon_0} \left[\frac{1}{\sqrt{a^2+r^2}} - \frac{1}{r} \right], \tag{53}$$

which is shown in figure 2 (right).

The ratio of the two is plotted in figure 3, from which it is seen that the multipole approximation becomes increasingly more precise as r becomes larger.

Exercise **4.4** Multipole expansion II

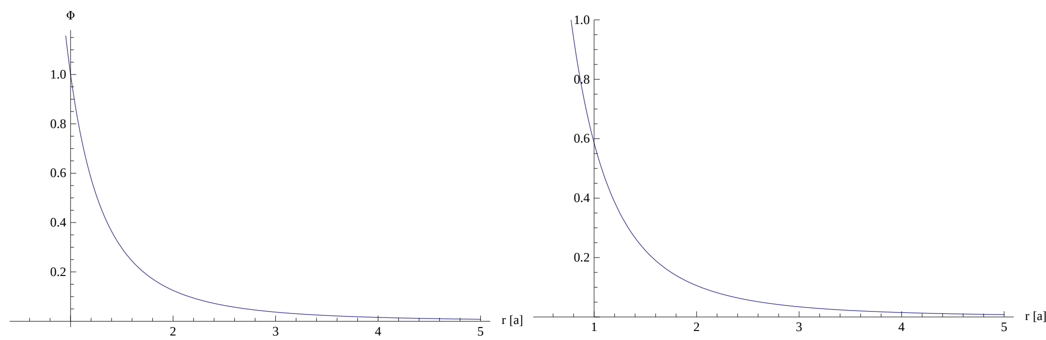

Figure 2: Potentials in units of $-\frac{q}{4\pi\epsilon_0 a}$, for the multipole expansion (left) and the exact calculation (right).

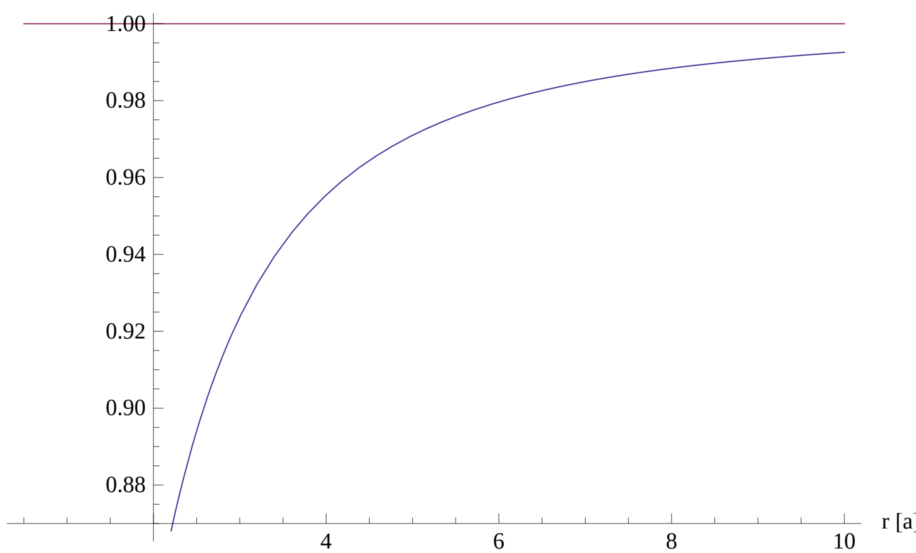

Figure 3: Ratio of the two potentials shown in figure 2. It is seen that the multipole approximation becomes increasingly more precise as r becomes larger.

(a) Prove the following theorem: For an arbitrary charge distribution $\rho(\vec{x})$ the values of the $(2l+1)$ moments of the first non vanishing multipole are independent of the origin of the coordinate axes, but the values of all higher multipole moments do in general depend on the choice of origin. (The different moments q_{lm} for fixed l depend, of course, on the orientation of the axes.)

(b) A charge distribution has multipole moments $q, \vec{p}, Q_{ij}, ...$ with respect to one set of coordinate axes and moments $q', \vec{p}', Q'_{ij}, ...$ with respect to another set whose axes are parallel to the first, but whose origin is located at the point $\vec{R} = (X, Y, Z)$ relative to the first. Determine explicitly the connections between the monopole, dipole and quadrupole moments in the two coordinate frames.

(c) If $q \neq 0$ can $\vec{R}$ be found so that $\vec{p}' = 0$? If $q \neq 0, \vec{p} \neq 0$, or at least $p \neq 0$, can $\vec{R}$ be found so that $Q_{ij} = 0$?

(a) The theorem is reformulated as follows:

Consider the (l'th) multipole moment coefficients (the multipole moment tensor in Cartesian coordinates), in a generalized form, ignoring the subtraction of the trace and the factor of 3:

$$Q_{ijk...l} = \int \rho(\vec{x}) x_i x_j x_k ... x_l \; d^3x. \tag{54}$$

Then $Q_{ijk...l}$ is translational invariant; i.e. invariant under coordinate transformations of the type: $x_i \mapsto x'_i = x_i - \alpha_i$, *iff* the l'th multipole coefficients are the first non vanishing ones.

Proof: The multipole moment tensor can be written in this new coordinate system (using for short $\rho' = \rho(\vec{x}' + \vec{\alpha})$):

$$
\begin{aligned}
Q_{ijk...l} = \int \rho'(x'_i + \alpha_i)(x'_j + \alpha_j)(x'_k + \alpha_k)...(x'_l + \alpha_l)\ d^3x = \\
\int \rho' x'_i x'_j ... x'_l\ d^3x + \\
\alpha_i \int \rho' x'_j ... x'_l\ d^3x + \\
\alpha_j \int \rho' x'_i x'_k ... x'_l\ d^3x + \\
\vdots \\
\alpha_l \int \rho' x'_i x'_j ... x'_{l-1}\ d^3x + \\
\alpha_i \alpha_j \int \rho' x'_k x'_{k+1} ... x'_l\ d^3x + \\
\alpha_i \alpha_j \alpha_k \int \rho' x'_{k+1} ... x'_l\ d^3x + \\
\vdots \\
\alpha_i \alpha_j ... \alpha_l \int \rho'\ d^3x = \\
Q'_{ijk...l} + \alpha_i Q'_{jk...l} + \alpha_j Q'_{ik...l} + ...
\end{aligned}
\tag{55}
$$

In the last step, it is realized that all terms are multipole moments in the new coordinate system. Consider first the case where all lower order multipole moments are zero. Clearly this implies that $Q_{ijk...l} = Q'_{ijk...l}$, and the theorem is proven one way. Consider on the other hand the situation where symmetry and hence $Q_{ijk...l} = Q'_{ijk...l}$ is given. The only way for this to be true for arbitrary ρ' and α_i, is if all lower terms are individually zero. The theorem is thus proved both ways.

The exercise makes the specific statement that: *"...the values of all higher multipole moments do in general depend on the choice of origin"*. This is clearly the case from the above statements, as multipole moments, which are not 'the first', will have extra terms in the new coordinate system, dependent of α_i.

(b) As a monopole has $l = 0$, a dipole has $l = 1$ and a quadrupole has $l = 2$, the direct calculation of the connection between these in the two coordinate systems is done by applying directly the expansion in equation (55).

Monopole (q):

$$q = \int \rho d^3x = \int \rho' d^3x = q'. \tag{56}$$

Dipole (p_i):

$$p_i = \int \rho x_i d^3x = \int \rho'(x_i' + \alpha_i) d^3x = \int \rho' x_i' d^3x + \alpha_i \int \rho' d^3x = p_i' + \alpha_i q'. \tag{57}$$

Quadrupole (Q_{ij}):

$$\begin{aligned} Q_{ij} = \int \rho x_i x_j d^3x = \int \rho'(x_i' + \alpha_i)(x_j' + \alpha_j)\ d^3x = \\ \int \rho' x_i' x_j'\ d^3x + \alpha_i \int \rho' x_j'\ d^3x + \alpha_j \int \rho' x_i'\ d^3x + \alpha_i \alpha_j \int \rho'\ d^3x = \\ Q_{ij}' + \alpha_j p_i' + \alpha_i p_j' + \alpha_i \alpha_j q'. \end{aligned} \tag{58}$$

(c) Using equation (57), it is seen that, if $p_i' = 0$ and $q \neq 0$ then:

$$\alpha_i = \frac{p_i}{q}. \tag{59}$$

If $q = 0$ and $p_i \neq 0$ then $p_i = p_i'$ and (by equation (58)):

$$\begin{aligned} Q_{ij}' = & Q_{ij} - \alpha_j p_i - \alpha_i p_j = 0 \Leftrightarrow \\ & Q_{ij} = \alpha_j p_i + \alpha_i p_j, \end{aligned} \tag{60}$$

which fully specifies α_i.

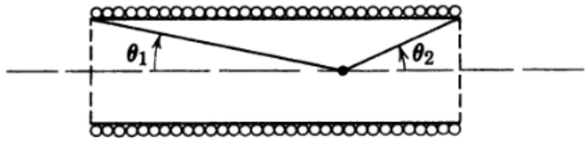

Figure 4: The solenoid related to problem 5.3.

Chapter 5

From the fifth chapter the exercises 5.3 and 5.7 are solved.

Exercise **5.3** Finite solenoid

A right circular solenoid of finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit $NL \to \infty$ is:

$$B_z = \frac{\mu_0 NI}{2}(\cos(\theta_1) + \cos(\theta_2)) \tag{61}$$

where the angles are defined in the figure (figure 4).

First the simple case of the magnetic field from a single current loop, as seen from a point on the centerline of the loop is treated. This situation is treated in quite some generality in Jackson 5.5, but the symmetries of this particular problem makes it easier to treat it again from the beginning. If the centerline is parallel to the z-axis then the line connecting the point and the intersection of the circular loop with the x-axis must be $|\vec{x}| = \sqrt{a^2 + z^2}$. The angle between the z-axis and this line is called θ. Now from the Biot-Savart law:

$$dB = \frac{\mu_0}{4\pi} I \frac{1}{|\vec{x}|^2} dI = \frac{\mu_0 I}{4\pi} \frac{1}{a^2 + z^2} dI, \tag{62}$$

and therefore (using $\sin(\theta) = \frac{a}{\sqrt{a^2+z^2}}$):

$$dB_z = dB \sin(\theta) = \frac{\mu_0 I}{4\pi} \frac{a}{(a^2 + z^2)^{3/2}} dI, \tag{63}$$

making the field (still only for a single loop):

$$B_z(\text{one loop}) = \oint \frac{\mu_0 I}{4\pi} \frac{a}{(a^2 + z^2)^{3/2}} dI = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}. \tag{64}$$

The limit given in the assignment, suggests that contributions from several loops should be integrated. As the loop density N is known the sought magnetic induction is found by integration over the whole coil. To end up with results in the same coordinate system as figure 4, the origin is set at the point on the z-axis where the two lines meet (with positive direction pointing left), and thus:

$$B_z = \frac{a^2\mu I N}{2}\left[\int_0^{z1} \frac{1}{(a^2+z^2)^{3/2}}\,dz + \int_{-z2}^{0} \frac{1}{(a^2+z^2)^{3/2}}\,dz\right]. \tag{65}$$

The integral is looked up, and it is used that $z_i = \cos(\theta_i)\sqrt{a^2+z_i^2}$, $i = \{1, 2\}$ resultantly:

$$B_z = \frac{a^2\mu_0 N I}{2}\left[\frac{z_1}{a^2\sqrt{a^2+z_1^2}} + \frac{z_2}{a^2\sqrt{a^2+z_2^2}}\right] = \frac{\mu_0 N I}{2}[\cos(\theta_1)+\cos(\theta_2)]. \tag{66}$$

Exercise **5.7** Helmholtz coils

A compact circular coil of radius a carrying a current I (perhaps N turns, each with current I/N), lies in the $x-y$ plane with its center at the origin.

(a) By elementary means [Eq. 5.4] find the magnetic induction at any point on the z-axis.

(b) An identical coil with the same magnitude and sense of the current is located on the same axis, parallel to, and a distance b above, the first coil. With the coordinate origin relocated at the point midway between the centers of the two coils, determine the magnetic induction on the axis near the origin as an expansion in powers of z, up to z^4 inclusive:

$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8} + ...\right], \tag{67}$$

where $d^2 = a^2 + b^2/4$.

(c) Show that, off-axis near the origin, the axial and radial components, correct to second order in the coordinates, take the form:

$$B_z = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right);\ B_\rho = -\sigma_2 z\rho. \tag{68}$$

(d) For the two coils in part **(b)** show that the magnetic induction on the z-axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part **(b)** by the formal substitution, $d \to |z|$.

(e) If $b = a$, the two coils are known as a pair of Helmholtz coils. For this choice of geometry the second terms in the expansions of parts **(b)** and **(d)** are absent ($\sigma_2 = 0$ in part **(c)**). The field near the origin is then very uniform. What is the maximum permitted value of $|z|/a$ if the axial field is to be uniform to one part in 10^4, one part in 10^2?

(a) The result from equation (64) (exercise 5.1) is reused:

$$B_z(\text{one loop}) = \oint \frac{\mu_0 I}{4\pi} \frac{a}{(a^2+z^2)^{3/2}} dI = \frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}. \tag{69}$$

(b) The principle of superposition is again used, but this time the magnetic field is the sum of the contributions from the two loops. In the coordinate system with origin in the middle of the two loops, distanced b from each other, the magnetic induction is clearly:

$$B_z = \frac{\mu_0 I a^2}{2}\left[\frac{1}{(a^2+(z-\frac{b}{2})^2)^{3/2}} + \frac{1}{(a^2+(z+\frac{b}{2})^2)^{3/2}}\right]. \tag{70}$$

A Taylor expansion of equation (70) around $z=0$, keeping only terms up to order z^4 is (using Mathematica):

$$B_z = \mu_0 I a^2 \left[\frac{8}{\left(4a^2+b^2\right)^{3/2}} - \frac{192\left(a^2-b^2\right)z^2}{\left(4a^2+b^2\right)^{7/2}} + \frac{1920\left(2a^4-6a^2b^2+b^4\right)z^4}{\left(4a^2+b^2\right)^{11/2}} + \mathcal{O}(z^5)\right], \tag{71}$$

which is readily brought to the desired form, using $d^2 = a^2 + b^2/4$:

$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8} + \mathcal{O}(z^5)\right]. \tag{72}$$

(c) This could in principle be done using the Biot-Savart law again, but since equation (72) can be used as a boundary value for the solution of the Laplace equation for this particular geometry, that is the approach taken. It is noted that the expression in equation (72) can be written as:

$$B_z(\rho=0) = -\nabla\Phi = \sigma_0 + \sigma_2 z^2, \tag{73}$$

with proper choice of σ_0 and σ_2. The magnetic field inside the cylinder spanned out by the two rings, is given by the Laplace equation:

$$\nabla^2\Phi = 0. \tag{74}$$

The solution to the Laplace equation in cylindrical coordinates is (Jackson equation 3.106):

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\int_0^{\infty} e^{-kz} J_m(k\rho)[A_m(k)\sin(m\phi) + B_m(k)\cos(m\phi)]\, dk, \tag{75}$$

where J_m is a Bessel function and A and B are coefficients given by the expressions in Jackson equation 3.109. As the problem exhibits azimuthal symmetry, the expression is greatly simplified (as only $m = 0$ contributes):

$$\Phi(\rho, z) = \int_0^\infty e^{-kz} J_0(k\rho) B_0(k)\ dk. \tag{76}$$

The Bessel function can be expanded using:

$$J_0 = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - ..., \tag{77}$$

making the potential:

$$\begin{aligned} \Phi(\rho, z) \approx \int_0^\infty e^{-kz} B_0(k) \left(1 - \frac{k^2\rho^2}{4}\right)\ dk = \\ \int_0^\infty e^{-kz} B_0(k)\ dk - \frac{\rho^2}{4} \int_0^\infty k^2 e^{-kz} B_0(k)\ dk = \\ \int_0^\infty e^{-kz} B_0(k)\ dk - \frac{\rho^2}{4}\frac{\partial^2}{\partial z^2} \int_0^\infty k^2 e^{-kz} B_0(k)\ dk \end{aligned} \tag{78}$$

where in the last step it is used that $k^2 e^{-kz} = \frac{\partial^2}{\partial z^2} e^{-kz}$. It is furthermore noticed that if $\rho = 0$, the second term vanishes. The first term is thus the boundary term $\Phi(0, z)$, and:

$$\Phi(\rho, z) = \Phi(0, z) - \frac{\rho^2}{4}\frac{\partial^2}{\partial z^2}\Phi(0, z). \tag{79}$$

Note that this result is general for the cylindrical geometry, and that both the coefficients A and B now have vanished. It is only now the specific configuration of the two coils enters the problem, by writing out the desired terms using equation (73) (and, of course, $\vec{B} = -\nabla\Phi$):

$$\begin{aligned} B_\rho =& \frac{\partial}{\partial\rho}\Phi(0, z) - \frac{\partial}{\partial\rho}\frac{\rho^2}{4}\frac{\partial^2}{\partial z^2}\Phi(0, z) = \\ & B_\rho(0, z) - \frac{\partial}{\partial\rho}\frac{\rho^2}{4}\frac{\partial}{\partial z} B_z(0, z) = \\ & 0 - \frac{2\rho}{4}(2\sigma_2 z) = -\rho\sigma_2 z. \\ B_z =& \frac{\partial}{\partial z}\Phi(0, z) - \frac{\partial}{\partial z}\frac{\rho^2}{4}\frac{\partial^2}{\partial z^2}\Phi(0, z) = \\ & (\sigma_0 + \sigma_2 z^2) - \frac{\rho^2}{4}\frac{\partial^2}{\partial z^2}(\sigma_0 + \sigma_2 z^2) = \\ & \sigma_1 + \sigma_2 z^2 - 2\frac{\rho^2}{4}\sigma_2 = \\ & \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right). \end{aligned} \tag{80}$$

(d) The Taylor expansion of equation (70) at $z = \infty$ becomes (again, using Mathematica):

$$\mu_0 I a^2 \left[\frac{1}{z^3} + \frac{3(b^2 - a^2)}{2z^5} + \frac{15(2a^4 - 6a^2b^2 + b^4)}{16z^7} + \mathcal{O}(1/z)^8 \right], \tag{81}$$

which is equivalent to equation (72) under the substitution $d \mapsto |z|$ (as seen by comparison).

(e) For $b = a$ equation (72) becomes:

$$B_z = \frac{\mu_0 I}{(\sqrt{5}/2)^3 a} \left[1 - \frac{144}{125} \frac{z^4}{a^4} \right]. \tag{82}$$

Since the second term in the parenthesis goes as z^4, it is taken as a correction term, and thus describes the uniformity of the field. Solving for uniformities of 10^{-4} and 10^{-2}, gives:

$$\frac{|z|}{a} = 0.097 \text{ and } \frac{|z|}{a} = 0.305 \tag{83}$$

respectively.

Chapter 6

From the sixth chapter the exercises 6.1 and 6.5 are solved.

Exercise **6.1**

In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at $t' = 0, \vec{x}' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$ (6.44). It may be initially surprising that in one or two dimensions, the disturbance possesses a ≫wake≪, even though the source is a ≫point≪ in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimensions(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

(a) Starting with the retarded solution to the three-dimensional wave equation (6.47), show that the source $f(\vec{x}', t') = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave:

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}, \tag{84}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function $[\Theta(\xi) = 0(1)$ if $\xi < (>) 0.]$.

(b) Show that a ≫sheet≪ source, equivalent to a point pulsed source at the origin in one space dimension produces a one-dimensional wave proportional to:

$$\Psi(x, t) = 2\pi c\Theta(ct - |x|). \tag{85}$$

The solution to the wave equation is given by:

$$\Psi(\vec{x}', t') = \int \frac{[f(\vec{x}', t')]_{ret}}{|\vec{x} - \vec{x}'|} \, d^3x', \tag{86}$$

where the $[\,]_{ret}$ indicates that $t' = t - |\vec{x} - \vec{x}'|/c$.

(a) Note that $|\vec{x} - \vec{x}'|^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$, and at $x' = y' = 0$, this becomes:

$$|\vec{x} - \vec{x}'| = \sqrt{\rho^2 + (z - z')^2}. \tag{87}$$

Performing the integration over x' and y' therefore yields:

$$\Psi = \int_{-\infty}^{\infty} \frac{\delta(t - \frac{1}{c}\sqrt{\rho^2 + (z - z')^2})}{\sqrt{\rho^2 + (z - z')^2}} \, dz'. \tag{88}$$

The integration variable is changed: $z' \mapsto u = z - z'$. To perform the integral it is used that, for a delta function of a function $f(u)$:

$$\delta(f(u)) = \sum_i \frac{1}{\left|\frac{\partial f}{\partial u}(u_i)\right| \delta(u - u_i)}, \tag{89}$$

where u_i are the roots of $f(u) = 0$. It is readily seen that $\frac{\partial f}{\partial u} = -\frac{1}{c}\frac{u}{\sqrt{\rho^2+u^2}}$, and that $f(u) = 0$ has two roots; $u_i = \pm\sqrt{c^2t^2 - \rho^2}$. To keep u from taking complex (unphysical) values, it must be demanded that $\rho \leq ct$. Since the two roots are related by $u_1 = -u_2$, Ψ is written as:

$$\begin{aligned} \Psi(x,y,t) = \int_{-\infty}^{\infty} \frac{c\sqrt{\rho^2 + u_1^2}}{u_1} \frac{\delta(u - u_1)}{\sqrt{\rho^2 + u^2}}\, du + \int_{-\infty}^{\infty} \frac{c\sqrt{\rho^2 + u_2^2}}{|u_2|} \frac{\delta(u + u_2)}{\sqrt{\rho^2 + u^2}}\, du = \\ \frac{c}{u_1} + \frac{c}{|u_2|} = \frac{2c}{u_1} = \frac{2c}{\sqrt{c^2t^2 - \rho^2}}. \end{aligned} \tag{90}$$

To enforce the demand that u should be real, the result is multiplied by the theta function $\Theta(ct - \rho)$, such that:

$$\Psi(x,y,t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}. \tag{91}$$

(b) By integrating over $\delta(x')$ in the source function, Ψ is this time:

$$\Psi = \int\int \frac{\delta(t - \frac{1}{c}\sqrt{x^2 + (y - y')^2 + (z - z')^2})}{\sqrt{x^2 + (y - y')^2 + (z - z')^2}}\, dy'dz'. \tag{92}$$

This simplifies greatly by going to polar (r, θ) coordinates in the $((y - y'), (z - z'))$-plane:

$$\Psi = \int_0^{2\pi} \int_0^{\infty} \frac{\delta(t - \frac{1}{c}\sqrt{x^2 + r^2})}{\sqrt{x^2 + r^2}}\, r dr d\theta = 2\pi \int_0^{\infty} \frac{\delta(t - \frac{1}{c}\sqrt{x^2 + r^2})}{\sqrt{x^2 + r^2}}\, r dr. \tag{93}$$

The delta function is treated as before, this time only the positive solution contributes, as r is positive definite, and in the end:

$$\Psi(x,t) = 2\pi c\theta(ct - |x|). \tag{94}$$

where the theta-function now enforces the demand that roots of $f(r) = 0$ should be real.

Exercise **6.5**

A localized electric charge distribution produces an electrostatic field, $\vec{E} = -\nabla\Phi$. Into this field is placed a small localized time-independent current density $\vec{J}(\vec{x})$, which generates a magnetic field $\vec{H}$.

(a) Show that the momentum of these electromagnetic fields, (6.117), can be transformed to:

$$\vec{P}_{field} = \frac{1}{c^2}\int \Phi\vec{J}\, d^3x, \tag{95}$$

provided the product $\Phi\vec{H}$ falls off rapidly enough at large distances. How rapidly is ≫rapidly enough≪?

(b) Assuming that the current distribution is localized to a region small compared to the scale of variation of the electric field, expand the electrostatic potential in a Taylor series and show that:

$$\vec{P}_{field} = \frac{1}{c^2}\vec{E}(0)\times\vec{m}, \tag{96}$$

where $\vec{E}(0)$ is the electric field at the current distribution and $\vec{m}$ is the magnetic moment, (5.54), caused by the current.

(c) Suppose the current distribution is placed instead in a *uniform* electric field $\vec{E}_0$ (filling all space). Show that, no matter how complicated is the localized $\vec{J}$, the result in part **(a)** is augmented by a surface integral contribution from infinity equal to minus one-third of the result of part **(b)**, yielding:

$$\vec{P}_{field} = \frac{2}{3c^2}\vec{E}_0\times\vec{m}. \tag{97}$$

Compare this result with that obtained by working directly with (6.117) and the considerations at the end of Section 5.6.

(a) Inserting $-\nabla\Phi$ for the electrostatic field the expression for the momentum (Jackson equation 6.117) becomes:

$$\vec{P}_{field} = \frac{1}{c^2}\int_V \vec{E}\times\vec{H}\, d^3x = -\frac{1}{c^2}\int_V \nabla\Phi\times\vec{H}\, d^3x. \tag{98}$$

This is rewritten using the vector identity: $\nabla\times(\psi\vec{a}) = \nabla\psi\times\vec{a} + \psi\nabla\times\vec{a}$. The momentum thus becomes a sum of two integrals:

$$\begin{aligned} c^2\vec{P}_{field} = &\int_V \Phi\nabla\times\vec{H}\, d^3x - \int_V \nabla\times\Phi\vec{H}\, d^3x = \\ &\int_V \Phi\nabla\times\vec{H}\, d^3x - \int_S \Phi\vec{H}\, d\vec{a}, \end{aligned} \tag{99}$$

where, in the last step, the second integral is rewritten as a surface integral. The full expression can thus be viewed as a leading term, minus a surface term. As the surface integral grows as distance squared, $\Phi\vec{H}$ must fall off faster than that, for the term to vanish. The remaining term is rewritten using Ampere's law, but as the current density is time-independent, $\partial_t\vec{D}=0$, so:

$$\vec{P}_{field} = \frac{1}{c^2}\int_V \Phi\vec{J}\, d^3x, \tag{100}$$

as desired.

(b) The magnetic moment (Jackson equation (5.52)) is:

$$\vec{m} = \frac{1}{2}\int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x. \tag{101}$$

First the electrostatic potential is expanded:

$$\Phi \approx \Phi(0) + \nabla\Phi(0)\cdot\vec{x} \tag{102}$$

The potential is chosen such that $\Phi(0)=0$, and by inserting in equation (100), an expression for $\vec{P}_{field}$ is obtained:

$$\vec{P}_{field} = -\frac{1}{c^2}\int_V (\vec{E(0)}\cdot\vec{x})\vec{J}\, d^3x. \tag{103}$$

In the following the arguments from Jackson p. 185 regarding a similar integral is followed closely. As $\vec{E(0)}$ does not depend on $\vec{x}$, it is moved outside the integral, and the ith component of the momentum can be written:

$$P^i_{field} = -\frac{1}{c^2}\sum_j E(0)_j \int_V J_i x'_j\, d^3x'. \tag{104}$$

From the identity in Jackson (5.52), using that $\vec{J}$ is localized and divergenceless, it is readily obtained that:

$$\int x'_i J_i\, d^3x' = -\frac{1}{2}\int x'_i J_j - x'_j J_i\, d^3x. \tag{105}$$

Therefore equation (104) becomes:

$$\begin{aligned} P^i_{field} =& \frac{1}{c^2}\frac{1}{2}\sum_j E(0)_j \int_V x'_i J_j - x'_j J_i\, d^3x = \\ & \frac{1}{c^2}\frac{1}{2}\sum_{j,k} \varepsilon_{ijk} E(0)_j \int_V (\vec{x}'\times\vec{J})_k\, d^3x = \\ & \frac{1}{c^2}\left(\vec{E}(0)\times\frac{1}{2}\int_V \vec{x}'\times\vec{J}\, d^3x'\right)_i . \end{aligned} \tag{106}$$

Making the full momentum vector:

$$\vec{P}_{field} = \frac{1}{c^2}\vec{E}(0)\times\vec{m}. \tag{107}$$

(c) The easiest way to obtain the desired result, is by working with Jackson equation (6.117) directly. As the electric field is uniform, it reduces to:

$$\vec{P}_{field} = \epsilon_0 \vec{E}_0 \times \int_V \vec{B}\ d^3x. \tag{108}$$

If all the current is contained within a sphere Jackson equation (5.62) applies, and:

$$\vec{P}_{field} = \frac{2}{3c^2} \vec{E}_0 \times \vec{m}, \tag{109}$$

directly.

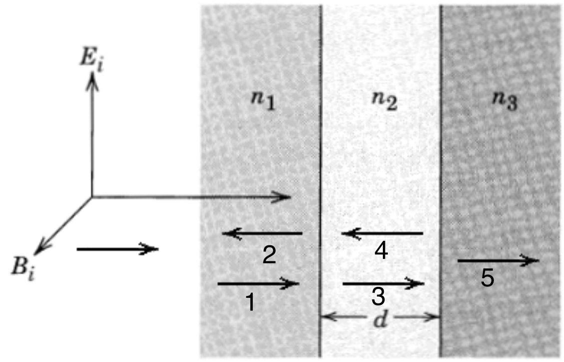

Figure 5: Sketch of the layer configuration with the wave numbering scheme drawn in.

Chapter 7

From the seventh chapter the exercises 7.2, 7.3, 7.5, 7.13 and 7.16 are solved.

Exercise **7.2** Reflection/refraction from layered interface

A plane wave is incident on a layered interface as shown in figure 5. The indices of refraction of the three non permeable media are n_1, n_2, n_3. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.

(a) Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for $n_1 = 1$, $n_2 = 2$, $n_3 = 3$; $n_1 = 3$, $n_2 = 2$, $n_3 = 1$; and $n_1 = 2$, $n_2 = 4$, $n_3 = 1$.

(b) The medium n_1 is part of an optical system (e.g., a lens); medium n_3 is air ($n_3 = 1$). It is desired to put an optical coating (medium n_2) on the surface so that there is no reflected wave for a frequency ω_0. What thickness and index of refraction n_2 are necessary?

(With a little inspiration from Griffiths' exercise 9.34)

(a) The materials are assumed linear and homogenous, and it is assumed that $\mu_1 = \mu_2 = \mu_3 = \mu_0$. Reflection and refraction are determined from boundary conditions at the surfaces. With the considered configuration, five waves is considered in total, as depicted on figure 5. Wave 1 traveling right in media I, wave 2 traveling left in media I, wave three traveling right in media II, wave four traveling left in media II and wave 5 traveling right in media III. Letting the interface between media I and II be situated at $z = 0$, then the interface between media II and III is at $z = d$. If the amplitudes of the 5 waves are written as $E_1, ..., E_5$, and the electric fields evolve as $\vec{E}_n = E_n \exp(i\vec{k}\vec{z})$, with positive z direction indicated on the picture. The interface between the first two layers gets a z coordinate of $z = 0$. The boundary conditions are (at $z = 0$):

$$\begin{aligned} E_1 + E_2 &= E_3 + E_4, \\ \sqrt{\frac{\epsilon_1}{\mu_0}}(E_1 - E_2) &= \sqrt{\frac{\epsilon_2}{\mu_0}}(E_3 - E_4), \end{aligned} \tag{110}$$

and (at $z = d$ (as $\vec{E}_5 = E_5 e^{i\vec{k_3}(\vec{z}-d)}$)):

$$\begin{aligned} E_3 e^{ik_2 d} + E_4 e^{-ik_2 d} &= E_5, \\ \sqrt{\frac{\epsilon_2}{\mu_0}}(E_3 e^{ik_2 d} - E_4 e^{-ik_2 d}) &= \sqrt{\frac{\epsilon_3}{\mu_0}} E_5. \end{aligned} \tag{111}$$

Using for short $\beta = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$ and $\alpha = \sqrt{\frac{\epsilon_3}{\epsilon_2}}$, the four equations becomes:

$$\begin{aligned} E_1 + E_2 &= E_3 + E_4, \\ (E_1 - E_2) &= \beta(E_3 - E_4), \end{aligned} \tag{112}$$

and (at $z = d$):

$$\begin{aligned} E_3 e^{ik_2 d} + E_4 e^{-ik_2 d} &= E_5, \\ E_3 e^{ik_2 d} - E_4 e^{-ik_2 d} &= \alpha E_5. \end{aligned} \tag{113}$$

Adding and subtracting the equations pairwise gives:

$$\begin{aligned} 2E_1 &= (1 + \beta)E_3 + (1 - \beta)E_4, \\ 2E_2 &= (1 - \beta)E_3 + (1 + \beta)E_4, \\ 2E_3 e^{ik_2 d} &= (1 + \alpha)E_5, \\ 2E_4 e^{ik_2 d} &= (1 - \alpha)E_5 \end{aligned} \tag{114}$$

It is then possible to write E_1 in terms of E_5:

$$E_1 = \frac{1}{4}\left[(1 + \alpha)(1 + \beta)e^{-ik_2 d} + (1 - \alpha)(1 - \beta)e^{ik_2 d}\right] E_5. \tag{115}$$

The expression in the square parenthesis is expanded in sines and cosines (and reduced):

$$E_1 = \frac{1}{2}\left[(\alpha\beta + 1)\cos(k_2 d) - (\alpha + \beta)i\sin(k_2 d)\right] E_5. \tag{116}$$

We can do the same thing to write E_2 in terms of E_5 (to be used later):

$$E_2 = \frac{1}{2}\left[(1-\alpha\beta)\cos(k_2 d) - (\alpha-\beta)i\sin(k_2 d)\right]E_5, \tag{117}$$

and insert the above expression to get E_2 in terms of E_1:

$$E_2 = \frac{(1-\alpha\beta)\cos(k_2 d) - (\alpha-\beta)i\sin(k_2 d)}{(1+\alpha\beta)\cos(k_2 d) - (\alpha+\beta)i\sin(k_2 d)}E_1. \tag{118}$$

The transmission coefficient is $T = \frac{\epsilon_3 v_3}{\epsilon_1 v_1}\left(\frac{E_5}{E_1}\right)^2 = \frac{n_3}{n_1}\left(\frac{E_5}{E_1}\right)^2$. This is calculated by first finding:

$$\begin{aligned}\left(\frac{E_1}{E_5}\right)^2 = \\ &\frac{1}{4}\left[(\alpha\beta+1)^2 - (\alpha\beta+1)^2\sin^2(k_2 d) + (\alpha+\beta)^2\sin^2(k_2 d)\right] = \\ &\frac{1}{4}\left[(\alpha\beta+1)^2 - (\alpha^2\beta^2+1+2\alpha\beta-\alpha^2-\beta^2-2\alpha\beta)\sin^2(k_2 d)\right] = \\ &\frac{1}{4}\left[(\alpha\beta+1)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)\right].\end{aligned} \tag{119}$$

By using $\alpha = n_3/n_2$, $\beta = n_2/n_1$ and $k_2 d = \omega n_2 d/c$, the transmission coefficient is:

$$T = \frac{4n_1 n_2^2 n_3}{n_2^2(n_1+n_3)^2 + (n_2^2-n_3^2)(n_2^2-n_1^2)\sin^2\left(\frac{n_2\omega d}{c}\right)}. \tag{120}$$

The reflection coefficient is then given by conservation of energy as: $R = 1 - T$.

It is seen that the expression in equation (120) in symmetric in exchange of n_1 and n_3. We therefore obtain the same result setting $n_1 = 1$, $n_2 = 2$ and $n_3 = 3$, as setting $n_1 = 3$, $n_2 = 2$ amd $n_3 = 1$ namely (using Mathematica):

$$T = \frac{96}{113 + 15\cos(4\omega d/c)}. \tag{121}$$

For $n_1 = 2$, $n_2 = 4$ amd $n_3 = 1$ we obtain:

$$T = \frac{32}{9(4 + 5\sin^2(4\omega d/c))}. \tag{122}$$

In figure (left) T and R from equation (121) are plotted, and in figure (right) T and R from equation (122) are plotted, all as a function of $\omega d/c$.

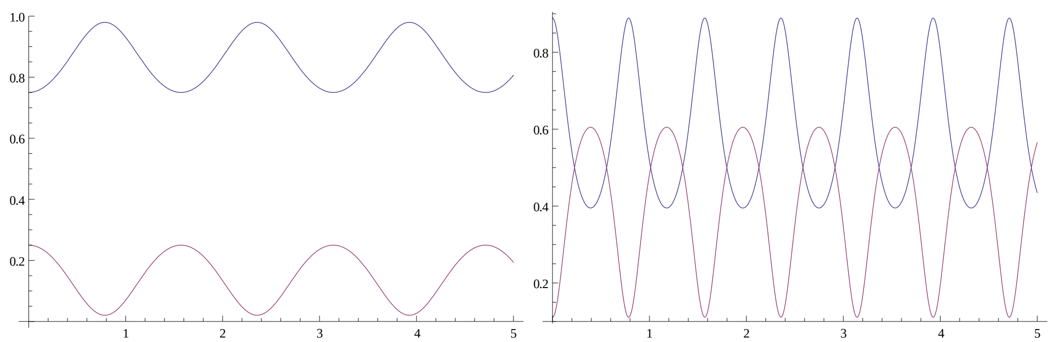

Figure 6: T (blue) and R (red) as a function of $\omega d/c$ given two sets of values of $n_1, ..., n_3$.

(b) Setting $n_3 = 1$, we get for R:

$$0 = R(\omega_0) = 1 - \frac{4n_1}{(1+n_1)^2 + \frac{(1-n_2^2)(n_1^2-n_2^2)\sin^2(n_2\omega d/c)}{n_2^2}}, \tag{123}$$

which implies that:

$$(n_1 - 1)^2 n_2^2 + (1 - n_2^2)(n_1^2 - n_2^2)\sin^2(n_2\omega d/c) = 0. \tag{124}$$

An option for solving this, is to equate the two terms to zero separately. The first term yields:

$$(n_1 - 1)^2 n_2^2 = 0 \Leftrightarrow (n_1 n_2 - 1)^2 = 0 \Leftrightarrow n_1 n_2 = 1. \tag{125}$$

The second term implies that:

$$\sin^2(n_2\omega d/c) = 0 \Rightarrow d = \frac{2\pi k c}{\omega n_2} \tag{126}$$

where k is an integer.

Exercise **7.3**

Two plane semi-infinite slabs of the same uniform, isotropic, non permeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) of width d. A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with angle of incidence i. For linear polarization *both* parallel to *and* perpendicular to the plane of incidence:

(a) calculate the ratio of power transmitted into the second slab to the incident power and the ratio of reflected to incident power.

(b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelengths in the gap.

(a) As this exercise is very similar to the 7.2, I will adopt the same notation (and also assume $\mu = \mu_0$ in the medium). By making the substitutions $d' = d/\cos(r)$, the boundary conditions for $\vec{E}$ perpendicular to the interface, are (at $z = 0$):

$$\begin{aligned} E_1 + E_2 &= E_3 + E_4, \\ \sqrt{\frac{\epsilon_1}{\mu_0}}(E_1 - E_2)\cos(i) &= \sqrt{\frac{\epsilon_2}{\mu_0}}(E_3 - E_4)\cos(r), \end{aligned} \tag{127}$$

and (at $z = d$):

$$\begin{aligned} E_3 e^{ik_2 d'} + E_4 e^{-ik_2 d'} &= E_5 e^{ik_1 d'}, \\ \sqrt{\frac{\epsilon_2}{\mu_0}}(E_3 e^{ik_2 d'} - E_4 e^{-ik_2 d'})\cos(r) &= \sqrt{\frac{\epsilon_1}{\mu_0}} E_5 e^{ik_1 d'} \cos(i). \end{aligned} \tag{128}$$

Using $\alpha = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \frac{\cos(i)}{\cos(r)} = n \frac{\cos(i)}{\cos(r)}$ and $\beta = \alpha^{-1}$, the four equations are equivalent to those already solved in the previous exercise up to $d \to d'$, and equation (119) is reused. Now:

$$\begin{aligned} T =& \frac{\epsilon_3 v_3}{\epsilon_1 v_1}\left(\frac{E_5}{E_1}\right)^2 = \left(\frac{E_5}{E_1}\right)^2 = \\ & \frac{4}{(1+1)^2 - (1 - \alpha^2 - \beta^2)\sin^2(k_2 d')} = \\ & \frac{1}{1 - \left(2 - n^2 \frac{\cos^2(i)}{\cos^2(r)} - \frac{1}{n^2}\frac{\cos^2(r)}{\cos^2(i)}\right)\sin^2(k_2 d')}. \end{aligned} \tag{129}$$

The reflection coefficient can still be calculated from conservation of energy, as $R = 1 - T$.

(b) For $i > i_0$, the relation $\cos(r) = i\sqrt{\left(\frac{\sin(i)}{\sin(i_0)}\right)^2 - 1}$ holds. The $\sin^2$ in the denominator will become $\sinh^2$, ensuring an exponential attenuation along the z direction, as the wave travels parallel to the interface. Fixing $n = 2$, we have $i_0 = \pi/6$. In figure 7 T is plotted as a function of wavelengths in the gap (x), using $k_2 d' = \frac{2\pi\lambda x}{\lambda\cos(r)}$, and the above relation between $\cos(r)$ and $\sin(i)$. The transmission coefficient is plotted

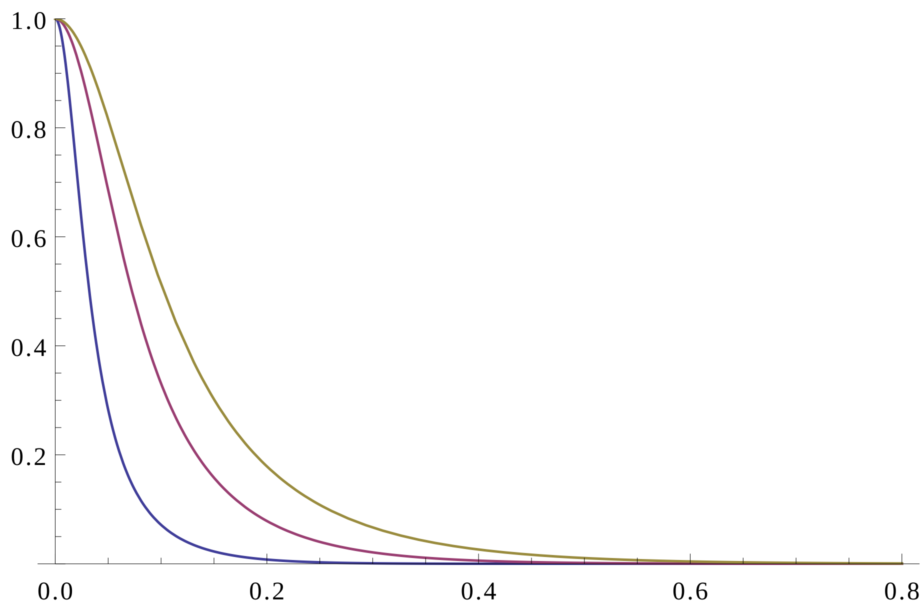

Figure 7: Transmission coefficient for $i = \pi/5$ (blue), $\pi/4$ (red) and $\pi/3$ (yellow), with $i_0 = \pi/6$.

for three different incident angles larger than i_0, $i = \pi/5$ (blue), $\pi/4$ (red) and $\pi/3$ (yellow). It is seen that (for fixed n) the smaller the incident angle is, the steeper the attenuation becomes.

Exercise **7.5**

A plane polarized electromagnetic wave $\vec{E} = \vec{E}_i e^{i\vec{k}\cdot\vec{x}-i\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma >> \omega\epsilon_0$) having a thickness D. Assuming that in space and in the conducting sheet $\mu/\mu_0 = \epsilon/\epsilon_0 = 1$, discuss the reflection and transmission of the incident wave.

(a) Show that the amplitudes of the reflected and transmitted waves, correct to the first order in $(\epsilon_0\omega/\sigma)^{\frac{1}{2}}$, are:

$$\begin{aligned}\frac{E_r}{E_i} &= \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda})+\gamma(1+e^{-2\lambda})},\\ \frac{E_t}{E_i} &= \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda})+\gamma(1+e^{-2\lambda})},\end{aligned} \tag{130}$$

where:

$$\begin{aligned}&\gamma = \sqrt{\frac{2\epsilon_0\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i),\\ &\lambda = (1-i)D/\delta\end{aligned} \tag{131}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the penetration depth.

(b) Verify that for zero thickness and infinite thickness you obtain the proper limiting results.

(c) Show that, except for sheets of very small thickness, the transmission coefficient is:

$$T = \frac{8(\mathfrak{Re}\,(\gamma)^2\, e^{-2D/\delta})}{1-2e^{-2D/\delta}\cos 2D/\delta + e^{-4D/\delta}}. \tag{132}$$

Sketch $\log T$ as a function of (D/δ), assuming $\mathfrak{Re}\,(\gamma) = 10^{-2}$.
Define ≫very small thickness≪.

(a) This is the same setup as before, only with a complex dielectric constant $\epsilon(\omega) = \epsilon_0 i\frac{\sigma}{\omega}$. The boundary conditions yield:

$$\begin{aligned}E_1 + E_2 &= E_3 + E_4,\\ (E_1 - E_2) &= \beta(E_3 - E_4),\\ E_3 e^{ik_2 D} + E_4 e^{-ik_2 D} &= E_5,\\ E_3 e^{ik_2 D} - E_4 e^{-ik_2 D} &= \alpha E_5,\end{aligned} \tag{133}$$

with: $\beta = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 - i\frac{\sigma}{\epsilon_0\omega}}$ and $\alpha = \beta^{-1}$. To obtain E_t/E_i (in my notation E_5/E_1), this is put directly into equation (116):

$$\frac{E_1}{E_5} = \cos(k_2 D) - (\alpha + \beta) i \sin(k_2 D). \tag{134}$$

Using $1 \ll \sigma/(\omega\epsilon_0)$ gives:

$$\begin{aligned} \beta &\approx \sqrt{i\frac{\sigma}{\epsilon_0\omega}} = (1+i)\sqrt{\frac{\sigma}{2\epsilon_0\omega}} = \frac{2}{\gamma}, \\ k_2 D &= \frac{2\omega D}{\gamma c} \approx i\lambda, \\ \alpha &\approx 0. \end{aligned} \tag{135}$$

Then equation (134) becomes:

$$\frac{E_1}{E_5} \approx \cos(i\lambda) - \frac{\beta}{2} i \sin(i\lambda) = \frac{\gamma(e^{\lambda} + e^{-\lambda}) + e^{\lambda} - e^{-\lambda}}{2\gamma}. \tag{136}$$

Inverting and multiplying numerator and denominator by $e^{-\lambda}$ gives the desired result:

$$\frac{E_5}{E_1} = \frac{2\gamma e^{-\lambda}}{\gamma(1 + e^{-2\lambda}) + 2(1 - e^{-2\lambda})}. \tag{137}$$

The second relation is obtained by plugging in to equation (118), following the same procedure:

$$\begin{aligned} \frac{E_2}{E_1} &= \frac{\frac{2}{\gamma} i \sin(i\lambda)}{2E_1/E_5} = -\frac{E_5}{2\gamma E_1}(e^{\lambda} - e^{-\lambda}) = \\ &- \frac{1}{2\gamma}\frac{E_5}{E_1}\frac{1 - e^{-2\lambda}}{e^{-\lambda}} = \frac{-(1 - e^{-2\lambda})}{(1 - e^{-2\lambda}) + \gamma(1 + e^{-2\lambda})}, \end{aligned} \tag{138}$$

as desired.

(b) The limiting result for zero thickness is: $D \to 0 \Rightarrow \lambda \to 0$ giving:

$$\frac{E_2}{E_1} \to 0 \text{ and } \frac{E_5}{E_1} \to 1 \tag{139}$$

The limiting result for infinite thickness is: $D \to \infty \Rightarrow \lambda \to \infty - \infty i$. Thus:

$$R \to \left(\frac{-1}{1+\gamma}\right) \text{ and } T \to 0. \tag{140}$$

The reflection coefficient only approaches 1 for a perfect conductor ($\gamma = 0$), as for an imperfect one there will always be attenuation in the material.

(c) Except for very small thickness, the denominator of E_5/E_1 will be a number of $\mathcal{O}(1)$ plus a very small number (of $\mathcal{O}(\gamma)$). The $\mathcal{O}(\gamma)$ term is thus ignored, and:

$$T \approx \left(\frac{2\gamma e^{-\lambda}}{1-e^{-2\lambda}}\right)^2. \tag{141}$$

Since the real and the imaginary part of both γ is of equal magnitude, its square is easily calculated as $|\gamma|^2 = 2\mathfrak{Re}\,(\gamma)^2$. The exponentials are written:

$$\left(e^{-\lambda}\right)^2 = \left(e^{iD/\delta}e^{-D/\delta}\right)^2 = e^{-2D/\delta}. \tag{142}$$

Then:

$$T \approx \frac{4|\gamma|^2 e^{-2D/\delta}}{1+e^{-4D/\delta}-2\mathfrak{Re}\left(e^{-2\lambda}\right)} = \frac{8\mathfrak{Re}\,(\gamma)^2\,e^{-2D/\delta}}{1-2\cos(2D/\delta)e^{-2D/\delta}+e^{-4D/\delta}}. \tag{143}$$

This approximation breaks down when the to terms of the denominator becomes roughly equal. The terms are Taylor expanded for small λ, and the term of $\mathcal{O}(\lambda\gamma)$ is dropped:

$$(1-e^{-2\lambda}) = \gamma(1+e^{-2\lambda}) \Rightarrow 2\lambda = 2\gamma - 2\lambda\gamma \Leftrightarrow D/\delta = \frac{\omega\delta}{c}, \tag{144}$$

meaning that »very small thickness« means $D < \omega\delta^2/c$.

Exercise **7.13**

A stylized model of the ionosphere is a medium described by the dielectric constant (7.59). Consider the earth with such a medium beginning suddenly at a height h and extending to infinity. For waves with polarization both perpendicular to the plane of incidence (from a horizontal antenna) and in the plane of incidence (from a verical antenna),

(a) show from Fresnel's equations for reflection and refraction that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is total reflection back toward the earth.

(b) A radio amateur operating at a wavelength of 21 meters in the early evening finds that she can recieve distant stations located more that 1000 km away, but none closer. Assuming that the signals are being reflected from the F layer of the ionosphere at an effective height of 300 km, calculate the electron density. Compare with the known maximum and minimum F later dendities of $\approx 2\times10^{12}\text{m}^{-3}$ in the daytime and $\approx (2-4)\times10^{11}\text{m}^{-3}$ at night.

(a) Inserting in Fresnel's equations (7.39) and (7.41) with $n = 1$ and $n' = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$ gives for both polarizations (Mathematica):

$$i = \text{ArcSin}\left(\frac{\sqrt{\omega^2 - \omega_p^2}}{\omega}\right). \tag{145}$$

(b) For the radio amateur to recieve a radio station from distance d away with a height h to the F layer, the incident angle should be:

$$\sin(i) = \frac{d}{\sqrt{d^2 + 4h^2}}. \tag{146}$$

Equating the two angles gives the plasma frequency:

$$\omega_p^2 = \omega\left(1 - \frac{d^2}{d^2 + 4h^2}\right), \tag{147}$$

which in turn gives the electron density (Jackson eq. 7.60):

$$NZ = \frac{\omega_p^2 \epsilon_0 m_e}{e^2} = 4\pi^2 \frac{c^2 \epsilon_0 m_e}{\lambda^2 e^2}\left(1 - \frac{d^2}{d^2 + 4h^2}\right). \tag{148}$$

The lump of natural constants gives $\frac{4\pi^2 m_e \epsilon_0 c^2}{e^2} = 1.11485 \cdot 10^{15}/\text{m}$. Plugging in numbers for the rest, gives $NZ = 6.69 \cdot 10^{11}\text{m}^{-3}$ in good agreement with the quoted numbers.

Exercise **7.16**

Plane waves propagate in a homogeneous, nonpermeable, but *anisotropic* dielectric. The dielectric is characterized by a tensor ϵ_{ij}, but if coordinate axes are chosen as the pricipal axes, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i (i = 1, 2, 3)$, where ϵ_i are the eigenvalues of the matrix ϵ_{ij}.

(a) Show that plane waves with frequency ω and wave vector $\vec{k}$ must satisfy:

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

(b) Show that for a given wave vector $\vec{k} = k\vec{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation:

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principal velocity, and n_i is the component if $\vec{n}$ along the ith principal axis.

(c) Show that $\vec{D}_a \cdot \vec{D}_b = 0$, where $\vec{D}_a, \vec{D}_b$ are the displacements associated with the two modes of propagation.

(a) Starting from Faraday's law, substituting $\nabla \to i\vec{k}$ and $\partial_t \to -i\omega$:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow i\vec{k} \times \vec{E} = i\omega\mu_0\vec{H} \Leftrightarrow \vec{k} \times (\vec{k} \times \vec{E}) = \omega\mu_0(\vec{k} \times \vec{H}). \tag{149}$$

Substituting in the Ampere-Maxwell equation in matter: $\nabla \times \vec{H} = \partial_t\vec{D} \Rightarrow i\vec{k} \times \vec{H} = -i\omega\vec{D}$ gives the desired result:

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0\omega^2\vec{D}. \tag{150}$$

(b) The result from the equation (150) is expanded using the vector triple product rule, and written in tensor notation:

$$\begin{gathered}(\vec{k} \cdot \vec{E})\vec{k} - (\vec{k} \cdot \vec{k})\vec{E} + \mu_0\omega^2\vec{D} = 0 \Leftrightarrow \\ (n_i n_j - \delta_{ij})E_i + \frac{v^2}{v_i^2}\delta_{ij}E_i = 0\end{gathered} \tag{151}$$

Taking the determinant and rearranging using $n_1^2 + n_2^2 + n_3^2 = 1$ (all using Mathematica) yields:

$$v^2(n_1^2(v^2 - v_2^2)(v^2 - v_3^2) + n_2^2(v^2 - v_1^2)(v^2 - v_3^2) + n_3^2(v^2 - v_1^2)(v^2 - v_2^2)) = 0, \tag{152}$$

which has a trivial solution $v = 0$, and by dividing by $\Pi_i(v^2 - v_i^2)$ the desired Fresnel equations are obtained:

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0. \tag{153}$$

(c) Starting from equation (151), with $A_{ij} = n_i n_j - \delta_{ij}$ and $B = \delta_{ij}\mu_0\epsilon_i$, we have for propagation modes a and b:

$$(A_{ij} + v_a^2 B_{ij})E_i^a = 0 \text{ and } (A_{ij} + v_b^2 B_{ij})E_i^b = 0. \tag{154}$$

Multiplying those equations by E_i^b and E_i^a respectively, and subtracting one from the other, gives:

$$E_i^b(v_a^2 - v_b^2)B_{ij}E_i^a = 0. \tag{155}$$

As $v_a \neq v_b$ this implies $E_i^b B_{ij} E_i^a = 0$. To show that the desired result is true, we however need $E_i^b (B_{ij})^2 E_i^a = 0$ to hold. This is true from the following argument: Since $B_i j \propto A_{ij}$ from above, and $(A_{ij})^2 = -A_{ij}$ (using again $n_1^2 + n_2^2 + n_3^2 = 1$), the desired substitution can be made, and $\vec{D}_a \cdot \vec{D}_b = 0$.

Chapter 8

From the eighth chapter the exercise 8.1 is solved.

Exercise **8.1** Excellent conductor

Consider the electric and magnetic fields in the surface region of an excellent conductor in the aproximation of Section 8.1, where the skin depth is very small compared to the radii of curvature of the surface or the scale of significant spatial variation of the fields just outside.

(a) For a single frequency component, show that the magnetic field $\vec{H}$ and the current density $\vec{J}$ are such that the time-averaged force per unit area of the surface from the conduction current, is given by:

$$\vec{f} = -\vec{n}\frac{\mu_c}{4}\left|\vec{H}_\parallel\right|^2, \tag{156}$$

where $\vec{H}_\parallel$ is the peak parallel component of magnetic field at the surface μ_c is the magnetic permeabillity of the conductor, and $\vec{n}$ is the outward normal at the surface.

(b) Repeat **(a)** but for an ideal conductor. Do you get the same result?

(c) Assume that the fields are a superposition of different frequencies (all high enough that the approximation still hold). Show that the time-averaged force takes the same form as in **(a)** with $\left|\vec{H}_\parallel\right|^2$ replaced by $2\left\langle\left|\vec{H}_\parallel\right|^2\right\rangle$, where the angle brackets $\langle ... \rangle$ mean time average.

(a) The force on a current $\vec{J}$ from a magnetic field $\vec{B}$ is $\vec{F} = \int_V \vec{J}\times\vec{B}\, d^3x$. Hence the force per unit area is:

$$\frac{d\vec{F}}{d\vec{A}} = -\int \vec{J}\times\vec{B}\, d\xi = \sigma\mu\int \vec{E}\times\vec{H}\, d\xi, \tag{157}$$

using Ohm's law and the definition of the $\vec{H}$ field. The overall sign changes since ξ is directed opposite to the normal. Following the discussion of time averaged Poynting vectors in Jackson p. 258-265, it is clear that the time average of the above quantity is:

$$\vec{f} = -\frac{1}{2}\sigma\mu_c\int_0^\infty \mathfrak{Re}(\vec{E}\times\vec{H}^*)d\xi \tag{158}$$

We now substitute in the solutions for $\vec{H}$ and $\vec{E}$ inside the conductor (Jackson eq. 8.9 and 8.10):

$$\vec{f} = -\sqrt{\frac{\mu_c^3\sigma\omega}{8}}((\vec{n}\times\vec{H}_\parallel)\times\vec{H}_\parallel)\int_0^\infty \mathfrak{Re}\left((1-i)e^{-\xi/\delta}e^{i\xi/\delta}e^{-\xi/\delta}e^{-i\xi/\delta}\right)d\xi. \tag{159}$$

The triple product in the paranthesis becomes:

$$(\vec{n} \times \vec{H}_{\|}) \times \vec{H}_{\|} = -\vec{H}_{\|}(\vec{n} \times \vec{H}_{\|}) = -\left(\left|\vec{H}_{\|}\right|^2 \vec{n} - (\vec{H}_{\|} \cdot \vec{n})\vec{H}_{\|}\right) = -\left|\vec{H}_{\|}\right|^2 \vec{n}. \quad (160)$$

And the integral:

$$\int_0^{\infty} \mathfrak{Re}\left((1-i)e^{-\xi/\delta}e^{i\xi/\delta}e^{-\xi/\delta}e^{-i\xi/\delta}\right) d\xi = \int_0^{\infty} e^{2\xi/\delta} d\xi = -\frac{\delta}{2}. \quad (161)$$

Putting it all together:

$$\begin{aligned} \vec{f} &= -\sqrt{\frac{\mu_c^3 \sigma \omega}{8}} \left|\vec{H}_{\|}\right|^2 \vec{n} \sqrt{\frac{1}{2\mu_c \omega \sigma}} \\ &= -\vec{n}\frac{\mu_c}{4}\left|\vec{H}_{\|}\right|^2. \end{aligned} \quad (162)$$

(b) If the conductor is perfect, we do not have any current inside the conductor. In other words, we only have a surface current $\vec{K} = \hat{n} \times \vec{H}$ (and no integral). Thus:

$$\frac{d\vec{F}}{d\vec{A}} = \frac{1}{2}\vec{K} \times \vec{B}^* = \frac{1}{2}(\hat{n} \times \vec{H}) \times \mu \vec{H}^* = -\frac{\mu}{2}\hat{n}\left|\vec{H}_{\|}\right|^2 \quad (163)$$

(c) From Jackson p. 264. the time averaging gives:

$$\left\langle |\vec{H}|^2 \right\rangle = \frac{1}{2}\mathfrak{Re}(\vec{H} \cdot \vec{H}^*). \quad (164)$$

But now the fields are superpositions of several frequencies, and thus:

$$\left\langle |\vec{H}|^2 \right\rangle = \frac{1}{2}\mathfrak{Re}\left(\sum_{i,j} H_i H_j^* \left\langle e^{-i(\omega_i - \omega_j)t} \right\rangle\right) = \frac{1}{2}|\vec{H}|^2. \quad (165)$$

The rest of the derivation is completely analogous to part **(a)**. Thus $\left|\vec{H}_{\|}\right|^2$ must be replaced by $2\left\langle \left|\vec{H}_{\|}\right|^2 \right\rangle$ in equation (162).

Chapter 9

From the ninth chapter the exercises 9.1, 9.8 and 9.16 are solved.

Exercise **9.1**

A common textbook example of a radiating system (see Problem 9.2) is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is not in the form of (9.1).

(a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\vec{x}, t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of (9.1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

(b) Consider a charge density $\rho(\vec{x}, t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier *series* expansion, show that it can be written as:

$$\rho(\vec{x}, t) = \rho_0(\vec{x}) + \sum_{n=1}^{\infty} \mathfrak{Re}\left[2\rho_n(\vec{x})e^{-in\omega_0 t}\right], \tag{166}$$

where:

$$\rho_n(\vec{x}) = \frac{1}{T}\int_0^T \rho(\vec{x}, t)e^{in\omega_0 t}dt. \tag{167}$$

This shows explicitly how to establish connection with (9.1).

(c) For a single charge q rotating about the origin in the $x - y$ plane in a circle of radius R at constant angular speed ω_0, calculate the $l = 0$ and $l = 1$ multipole moments by the methods of parts **(a)** and **(b)** and compare. In method **(b)** express the charge density $\rho_n(x)$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

(a) The charge density in the co-rotating coordinate system is in full generality:

$$\rho(r, \theta, \phi^*) = \rho(r, \theta, \phi - \omega t). \tag{168}$$

For the specific configuration in exercise (9.2) the charge density takes the form (with $q_i = \pm 1$, depending on what corner we are in):

$$\rho(r, \theta, \phi - \omega t) = \sum_{i=1}^{4} \frac{q_i}{r^2 \sin(\theta)}\delta(\theta - \theta_i)\delta(r - r_i)\delta(\phi - \phi_i), \tag{169}$$

but the following considerations can be done in general for any rigid, rotating charge distribution.

Inserting in the equation for the multipole moment (Jackson (4.3)) yields:

$$q_{lm}(t) = \int Y^*_{lm}(\theta', \phi' - \omega t) r'^l \rho(r', \theta', \phi' - \omega t) dr' d\phi' d\theta'. \tag{170}$$

Since the ϕ-dependence of a spherical harmonic is always in the form of a complex exponential, the time dependence factorizes:

$$q_{lm}(t) = \int Y^*_{lm}(\theta', \phi') r'^l \rho(r', \theta', \phi') dr' d\phi' d\theta' \exp(-im\omega t) = q^0_{lm} \exp(-im\omega t), \tag{171}$$

where q^0_{lm} is the fixed time multipole moment (at $t = 0$). For the sample case of equation (169) the fixed time multipole moment becomes:

$$q^0_{lm} = \sum_{i=1}^{4} r_i^l q_i Y^*_{lm}(\theta_i, \phi_i). \tag{172}$$

(b) Fourier series expansion of $\rho(\vec{x}, t)$ yields:

$$\rho(\vec{x}, t) = \sum_{n=-\infty}^{\infty} \rho_n(\vec{x}) \exp(-in\omega t), \text{ with } \rho_n(\vec{x}) = \frac{1}{T} \int_0^T \rho(\vec{x}, t) \exp(i\omega t) dt. \tag{173}$$

Separating the $n = 0$, $n < 0$ and $n > 0$ parts, and noting that $\rho_{-n} = \rho^*_n$ since physical quantities are always real:

$$\begin{aligned} \rho(\vec{x}, t) &= \rho_0 + \sum_{n=-\infty}^{-1} \rho_n(\vec{x}) \exp(-in\omega_0 t) + \sum_{n=1}^{\infty} \rho_n(\vec{x}) \exp(-in\omega_0 t) \\ &= \rho_0 + \sum_{n=1}^{\infty} (\rho^*_n \exp(in\omega_0 t) + \rho_n \exp(-in\omega_0 t)) \\ &= \rho_0 + \sum_{n=1}^{\infty} \mathfrak{Re} \left[2\rho_n \exp(-in\omega_0 t) \right], \end{aligned} \tag{174}$$

which is the desired result.

(c) First method **(a)**. The charge distribution in the co-moving coordinate system is:

$$\rho(r, \theta, \phi - \omega t) = q\delta(r - R)\delta(\cos(\theta))\delta(\phi). \tag{175}$$

Thus the fixed time multipole moment is:

$$\begin{aligned} q_{lm}^0 &= \int r'^l Y_{lm}^*(\theta',\phi')\rho(r,\theta',\phi')dr'd\theta'd\phi' \\ &= -\int r'^l Y_{lm}^*(\theta',\phi')\rho(r,\theta',\phi')\frac{1}{\sin(\theta)}dr'd\cos(\theta)d\phi' \\ &= -qR^l Y_{lm}^*(\pi/2,0). \end{aligned} \tag{176}$$

As the time dependent multipole moment is just $q_{lm}(t) = q_{lm}^0 \exp(-im\omega t)$, we have (using the table in Jackson p. 109):

$$\begin{aligned} q_{00}(t) &= -qY_{00}^*(\pi/2,0) = -q\frac{1}{\sqrt{4\pi}} \\ q_{11}(t) &= -qRY_{11}^*(\pi/2,0)\exp(-i\omega t) = qR\sqrt{\frac{3}{8\pi}}\exp(-i\omega t) \\ q_{10}(t) &= -qRY_{10}^*(\pi/2,0) = -qR\sqrt{\frac{3}{4\pi}}\cos(\pi/2) = 0 \\ q_{1-1}(t) &= -qRY_{1-1}^*(\pi/2,0)\exp(i\omega t) = -qR(-1)^1 Y_{11}(\pi/2,0)\exp(i\omega t) \\ &= -qR\sqrt{\frac{3}{8\pi}}\exp(i\omega t) \end{aligned} \tag{177}$$

With method **(b)** first ρ_n is calculated:

$$\begin{aligned} \rho_n &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \rho(\vec{x},t)\exp(in\omega_0 t)dt \\ &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \frac{q}{R^2}\delta(r-R)\delta(\cos(\theta))\delta(\phi-\omega_0 t)dt = \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos(\theta))\exp(in\phi). \end{aligned} \tag{178}$$

This is plugged in to the multipole moment equation:

$$\begin{aligned} q_{lm} &= \frac{q}{2\pi R^2}\int r^{2+l}Y_{lm}^*(\theta,\phi)\delta(r-R)\delta(\cos(\theta))\exp(in\phi)drd\cos\theta d\phi \\ &= -qR^l Y_{lm}^*(\pi/2,0) \text{ if } m=n, \text{ 0 otherwise,} \end{aligned} \tag{179}$$

which agrees with the above for $m = n$.

Higher orders are present as long as the spherical harmonic function is non-vanishing, *i.e.* $P_{lm}(0) \neq 0$

Exercise **9.8**

(a) Show that a classical oscillating dipole $\vec{p}$ with fields given by (9.18) radiates electromagnetic angular momentum to infinity at the rate:

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\mathfrak{Im}[\vec{p}^* \times \vec{p}]. \tag{180}$$

(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

(c) For a charge e rotating in the $x-y$ plane at a radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^3k^3a^2/6\pi\epsilon_0$. What about a charge oscillating along the z axis?

(d) What are the results corresponding to parts **(a)** and **(b)** for magnetic dipole radiation?

Hint: The electromagnetic angular momentum density comes from more than the transverse (radiation zone) components of the fields.

(a) The angular momentum density is:

$$\begin{aligned}\vec{m} &= \vec{r}\times\left(\frac{1}{2c^2}\vec{E}\times\vec{H}^*\right)\\ &= \frac{1}{2c^2}\left(\vec{E}(\vec{r}\cdot\vec{H}^*) - \vec{H}^*(\vec{r}\cdot\vec{E})\right) = \frac{1}{2c^2}\vec{H}^*(\vec{r}\cdot\vec{E}),\end{aligned} \tag{181}$$

as $\vec{r}\cdot\vec{H}^* = 0$. Meanwhile, using in general that:

$$\vec{a}\cdot((\vec{a}\times\vec{b})\times\vec{a}) = \vec{a}\cdot(-(\vec{a}\cdot\vec{b})\vec{a} + (\vec{a}\cdot\vec{a})\vec{b}) = -(\vec{a}\cdot\vec{b})a^2 + a^2(\vec{a}\cdot\vec{b}) = 0, \tag{182}$$

such that:

$$\begin{aligned}\vec{r}\cdot\vec{E} &= r\vec{n}\cdot\vec{E}\\ &= \frac{1}{4\pi\epsilon_0}\left(k^2\vec{n}\cdot((\vec{n}\times\vec{p})\times\vec{n})\exp(ikr) + \vec{n}\cdot(3\vec{n}(\vec{n}\cdot\vec{p}) - \vec{p}\left(\frac{1}{r^2}-\frac{ik}{r}\right)\exp(ikr))\right)\\ &= \frac{\exp(ikr)}{4\pi\epsilon_0}\left((3(\vec{n}\cdot\vec{p}) - (\vec{n}\cdot\vec{p}))\left(\frac{1}{r^2}-\frac{ik}{r}\right)\right)\\ &= \frac{\exp(ikr)}{2\pi\epsilon_0}(\vec{n}\cdot\vec{p})\left(\frac{1}{r^2}-\frac{ik}{r}\right).\end{aligned} \tag{183}$$

Inserting above gives:

$$\vec{m} = \frac{ik^3}{16\pi^2\epsilon_0 cr^2}\left(1+\frac{1}{k^2r^2}\right)(\vec{n}\cdot\vec{p})(\vec{n}\times\vec{p}^*), \tag{184}$$

which should be integrated over a sphere at radius r to give the result:

$$\frac{d\vec{L}}{dt} = \int \vec{m}\,d\Sigma. \tag{185}$$

(b) The radiated energy is simply the total power $P = \frac{c^2 Z_0 k^4}{12\pi}|\vec{p}|^2$, and the ratio is thus:

$$\frac{1}{P}\frac{d\vec{L}}{dt} = \frac{1}{c^2 Z_0 \epsilon_0 k}\frac{\mathfrak{Im}\left(\vec{p}\times\vec{p}^*\right)}{|\vec{p}|^2} = \frac{1}{\omega}\frac{\mathfrak{Im}\left(\vec{p}\times\vec{p}^*\right)}{|\vec{p}|^2}, \tag{186}$$

which is seen to vanish for high frequencies. Thus high frequencies favours radiation of energy over radiation of angular momentum.

(c) The charge density is:

$$\rho(\vec{x},t) = e\delta(x - a\cos(\omega t))\delta(y - a\sin(\omega t))\delta(x). \tag{187}$$

Inserting into Jackson eq. 9.17 gives:

$$\vec{p} = \int \vec{x}\rho d^3x = ea(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)). \tag{188}$$

By inserting a mock complex part $eai(\hat{y}\cos(\omega t) - \hat{x}\sin(\omega t))$, this is rewritten as:

$$\vec{p} = ea\mathfrak{Re}\left((\hat{x} + i\hat{y})\exp(-i\omega t)\right). \tag{189}$$

As only the magnitude of $d\vec{L}/dt$ is wanted, the time dependent part is omitted, and the rest is inserted in the answer from part **(a)**:

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\mathfrak{Im}\left((ea(\hat{x}+i\hat{y})^* \times ea(\hat{x}+i\hat{y}))\right) = \frac{e^2a^2k^3}{6\pi\epsilon_0}. \tag{190}$$

In the case of a charge oscillating on the z-axis, the charge distribution will contain delta functions $\delta(x)$ and $\delta(y)$. Therefore $\vec{p} \approx \hat{z}$, and the cross product in the last equation will be zero.

(c) The result in part **(a)** for dipole radiation can be obtained by substituting $\vec{p} \mapsto \vec{m}/c$ resulting in:

$$\frac{d\vec{L}}{dt} = \frac{k^3}{12\pi\epsilon_0 c^2}\mathfrak{Im}\left(\vec{m}^* \times \vec{m}\right). \tag{191}$$

The change in total emitted power is analogously $|\vec{p}|^2 \mapsto |\vec{m}|^2/c^2$, and thus the result in part **(b)** is unchanged up to the change $\vec{p} \mapsto \vec{m}$.

Exercise **9.16**

A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.

(a) The current in the rod is sinusoidal: $I(z) = I_0 \sin(kz)$. Writing this as the total current distribution is:

$$\vec{J}(\vec{x}) = I_0 \sin(kz)\delta(x)\delta(y)\hat{z}. \tag{192}$$

From Jackson eq. (9.8) the vector potential is:

$$\begin{aligned}
\lim_{kr\to\infty} \vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\frac{\exp ikr}{r}\int \vec{J}(\vec{x'})\exp(-ik\vec{n}\cdot\vec{x'})d^3x' \\
&= I_0\frac{\mu_0}{4\pi}\frac{\exp ikr}{r}\hat{z}\int_{-d/2}^{d/2}\sin(kz')\exp(-ikz'\cos(\theta))dz' \\
&= I_0\frac{\mu_0}{4\pi}\frac{\exp ikr}{r}\hat{z}\int_{-d/2}^{d/2}\sin(kz')\cos(kz'\cos(\theta)) + i\sin(kz')\sin(kz'\cos(\theta))dz'.
\end{aligned} \tag{193}$$

The first part of the integral is an odd function, and is therefore zero when integrated over the given interval. The remaining integral can be looked up and the vector potential becomes:

$$\begin{aligned}
\lim_{kr\to\infty} \vec{A}(\vec{x}) = I_0\frac{\mu_0}{4\pi}\frac{\exp ikr}{r}i\hat{z}\frac{1}{k-\cos^2(\theta)k}\cdot \\
(2\cos(\theta)\sin(d/2k)\cos(\cos(\theta)d/2k) - 2\cos(d/2k)\sin(\cos(\theta)d/2k))
\end{aligned} \tag{194}$$

Inserting $k = \frac{2\pi}{d}$ the first term in the parenthesis vanishes, and the rest of the expression is (using also $1-\cos^2(\theta) = \sin^2(\theta)$):

$$\lim_{kr\to\infty} \vec{A}(\vec{x}) = -I_0\frac{\mu_0}{4\pi^2}\frac{\exp(ikr)}{r}di\hat{z}\frac{\sin(\cos(\theta)\pi)}{\sin^2(\theta)}. \tag{195}$$

The dipole moment becomes (Jackson eq. 9.16):

$$\begin{aligned}
\vec{p} &= -\frac{4\pi}{i\mu_0\omega}\frac{r}{\exp(ikr)}\vec{A}(\vec{x}) \\
&= \frac{2I_0}{k^2c}\hat{z}\frac{\sin(\cos(\theta)\pi)}{\sin^2(\theta)}.
\end{aligned} \tag{196}$$

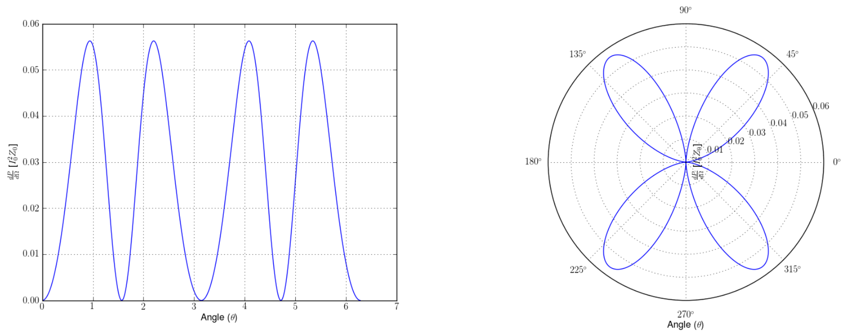

Figure 8: The power radiation distribution from exercise 9.16 in a regular (left) plot and a polar (right) plot.

And the angular distribution (Jackson eq. 9.23):

$$\frac{dP}{d\Omega} = \frac{I_0^2 Z_0}{8\pi^2} \frac{\sin^2(\cos(\theta)\pi)}{\sin^2(\theta)},, \tag{197}$$

which is seen in figure (left) as a regular plot, and much nicer in figure (right) as a polar plot.

(b) The total power radiated is:

$$\begin{aligned} P &= \frac{I_0^2 Z_0}{8\pi^2} \int_0^{2\pi} \int_{-1}^{1} \frac{\sin^2(\cos(\theta)\pi)}{\sin^2(\theta)} d\phi d\cos(\theta) \\ &= \frac{I_0^2 Z_0}{4\pi^2} \int_{-1}^{1} \frac{\sin^2(\pi x)}{1 - x^2} dx \approx 1.5572 \frac{I_0 Z_0}{4\pi}. \end{aligned} \tag{198}$$

And the radiation resistance is:

$$R_{rad} = \frac{P}{I_0^2/2} = \frac{1.5572 Z_0}{2\pi} = 93.39\Omega. \tag{199}$$